MUC-NOTE-COOL_THEORY-241

Stability analysis of the Study-II 2.75 m lattice

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9 April 2002

Abstract

We examine the stability of the Feasibility Study II 2.75 m lattice. The transmission of the lattice is determined as a function of momentum using ICOOL. We find that the transmission data agrees very well with a simple model of the solenoid lattice based on the multiplication of symplectic matrices. We also find good agreement with the Hill function analysis of Wang and Kim.

1 Introduction

In this report we look at the transmission properties of the Feasibility Study II (FS2) 2.75 m lattice [1]. This lattice is important since it is still the baseline bunching and cooling lattice for ongoing studies of the neutrino factory and no detailed study of its transmission properties have been presented yet. In addition, a closely related lattice cell has been proposed [2] for the MICE demonstration of ionization cooling.

A number of authors have presented theoretical analyses related to the range of parameters giving stable motion through solenoidal lattices. An analysis based on the eigenvalues of the Mathieu equation [3] gave predictions for the stable conditions that agreed precisely with the results of tracking studies using ICOOL. This analysis only applied to so-called FOFO lattices, where the axial component of the solenoid field on-axis was exactly sinusoidal. It was soon realized [4] that FOFO lattices had a very large field enhancement factor at the coils and were not practical in cooling channels.

A study of more general super-FOFO lattices that included additional harmonic content in the magnetic field was performed by Penn [5]. Using a thin lens model he was able to show that two regions of stable momentum values were available in these lattices. Theoretical calculations were compared to the results of beam moments calculations. Balbekov [4] has presented results for the resonance behavior of a very similar lattice, which has the same geometry, but slightly higher current density in the coils.

A detailed study of stability using recursive solutions of the Hill equation was made by Wang & Kim [6]. They derived expressions for a stability parameter and for determining the beta function. Although calculations were made by the authors for several experimental arrangements, the results were not checked by comparison with either moment equations or tracking. We do a detailed comparison here of the ICOOL results with the predictions of this theory.

2 Basic properties of the FS2 (1,1) lattice

The FS2 cooling channel used two different geometric lattices, first one with 2.75 m cell length and then a second with 1.65 m length. Each of these lattices were further subdivided into three regions where the geometry was the same, but the current density in the solenoid coils was tapered upwards. We use the shorthand (1,1) to refer to the 2.75 m lattice with the lowest set of current densities. (The results by Balbekov [4] referred to earlier correspond to the (1,2) lattice in FS2.) Each cell of the lattice contains two focusing solenoids and one coupling solenoid. The coil parameters are given in Table 1.

Table 1 FS2 (1,1) coil properties [1]

Type	Position	Length	Radius	Thickness	J
	[m]	[m]	[m]	[m]	A / mm^2
Focusing 1	0.175	0.167	0.330	0.175	75.20
Coupling	1.210	0.330	0.770	0.080	98.25
Focusing 2	2.408	0.167	0.330	0.175	75.20

In the table Position gives the axial distance from the beginning of the cell and Radius refers to the inner radius of the coil. The axial component of the solenoid field produced by these coils is shown in Fig. 1.

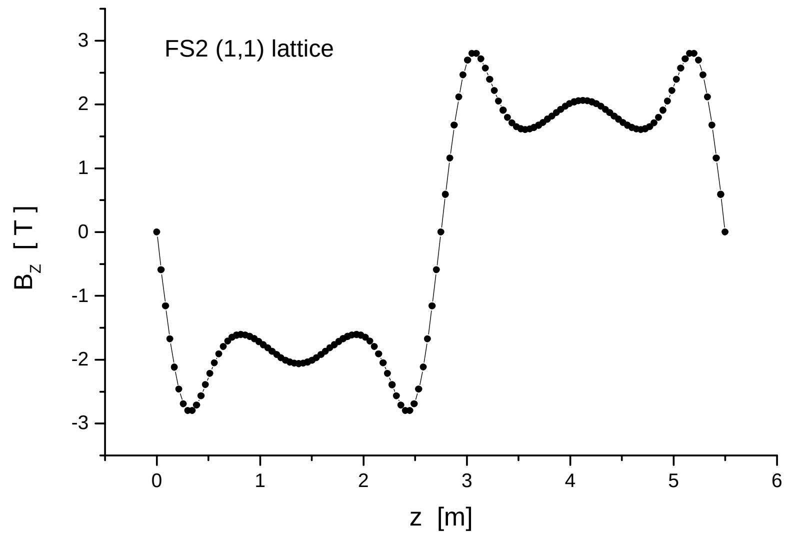

Figure 1. Two cells (one magnetic period) of the solenoidal field on-axis.

The figure shows one period of the magnetic field that includes one lattice cell with a set of coils given by Table 1 and a second lattice cell with the same geometry, but with the current density reversed in polarity. The peak value of the magnetic field occurs at a distance of 32 cm from the 0 field point. The integrated field over half a period (2.75 m) is 5.303 T m. The expectation value of the squared field is 3.988 T^2.

Although the field pattern is roughly sinusoidal, it has strong contributions from higher harmonics. The axial field can be described as

$$B(z) = \frac{B_o}{1.40} \sum_{odd\ n} c_n \sin n k z \qquad \textbf{(1)}$$

where

$$k = \frac{2\pi}{\lambda} \qquad \textbf{(2)}$$

and λ = 5.5 m is the period of the magnetic field. B_o = 2.813 T is the peak value of the field on-axis. The first ten non-vanishing coefficients of the Fourier decomposition of the field shown in Fig. 1 is given in Table 2.

Table 2 Relative harmonic content	
n	c_n / c_1
1	1.0000
3	0.4165
5	0.3446
7	0.1159
9	0.0227
11	-0.0081
13	-0.0100
15	-0.0068
17	-0.0038
19	-0.0022

This field has a rich harmonic content with particularly strong components around n = 3 and 5. The magnitude of the fundamental coefficient was $c_1 = 1.2329$.

We next want to consider the effects of the lattice on incident particles with different momenta. We take a gaussianly distributed initial beam with a normalized transverse emittance of 12 mm and 0 canonical angular momentum. This emittance is roughly the FS2 value that this lattice was designed to transmit. To begin with we take a fixed beam size of 42 mm *rms* and a transverse momentum of 30 MeV/c *rms* in x and y. Using ICOOL (v2.33) we set up a lattice consisting of 20 cells and with a radial aperture of 1 m. The transmission as a function of momentum is shown in Fig. 2.

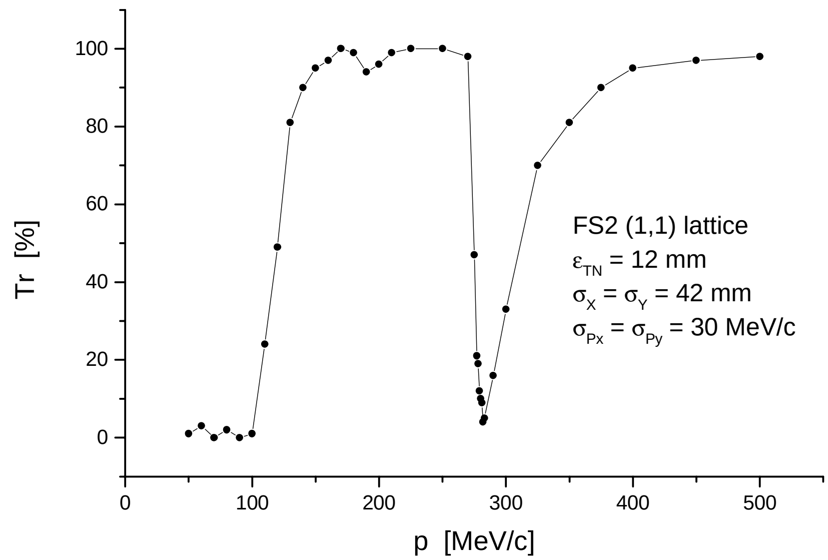

Figure 2. ICOOL calculation of the lattice transmission as a function of momentum.

The dominant features of this figure are the high momentum transmission band and a second narrower band of transmission, roughly between 110 and 280 MeV/c. Penn has defined [5] the high momentum band as region I and the narrower band as region II. It is this second band that was actually used in FS2. There is a very strong stop band centered at 282 MeV/c, which we will see in the following section is identified with the π resonance. There is essentially no transmission in this lattice below 100 MeV/c.

3 Location of the π resonance

It is useful to have a easy way to calculate *a priori* the value of the momentum where we expect the π resonance to occur. This resonance occurs when the betatron wavelength Λ of the particle equals the period λ of the magnetic field. In an earlier study [3] of FOFO lattices we found that

$$\Lambda \approx \frac{2\pi\sqrt{8}}{e}\frac{p}{B_o} \approx 59\frac{p}{B_o} \quad \textbf{(3)}$$

where the units used are {m, GeV/c, T}. This gives the location of the resonance at

$$p_\pi \approx \frac{B_o \lambda}{59} \tag{4}$$

or at 261 MeV/c. Another estimate of the location of the resonance in a super-FOFO lattice (not the same one) was given by Penn [5], who found that

$$p_\pi \approx \frac{B_o \lambda}{57} \tag{5}$$

which gives 271 MeV/c. A third estimate was given by Wang & Kim [6] for a general lattice, who found that

$$p_\pi \approx \frac{e \lambda}{4 \pi} \sqrt{<B^2>} \approx \frac{\lambda B_o}{59} \tag{6}$$

in excellent agreement with the other estimates.

In any case it is fairly clear that the stop band at 282 MeV/c in Fig. 2 is due to the π resonance. In that case we expect the 2π resonance to be around 140 MeV/c, where we see the low momentum cut off in transmission. It is interesting that the very weak drop in transmission around 200 MeV/c is near the $3\pi/2$ half-integer resonance.

4 Beta function for the lattice

One problem with the transmission study in Fig. 2 is that the initial beam distribution is not matched for all momenta because the matched beta function changes with momentum. To remedy this we determine the matched beta function for any given momentum. By matched we mean the value of the beta function and its first derivative are the same at the beginning and end of each cell. To do this a code was written in which an optimizer used the value of the beta function and its first derivative at the beginning of the cell as parameters. The merit function for the optimization was determined by comparing these quantities with their values at the end. To get the end values given the initial conditions we numerically solved the differential equation [7,8]

$$2 \beta \frac{d^2 \beta}{dz^2} - \left(\frac{d\beta}{dz} \right)^2 + 4 \beta^2 \kappa^2 - 4 = 0 \tag{7}$$

where

$$\kappa(z) = \frac{e\,B(z)}{2\,p} \tag{8}$$

which is formally identical to the well known result for quadrupole channels. Results for the minimum and maximum values of the beta function in each cell as a function of momentum are shown in Fig. 3.

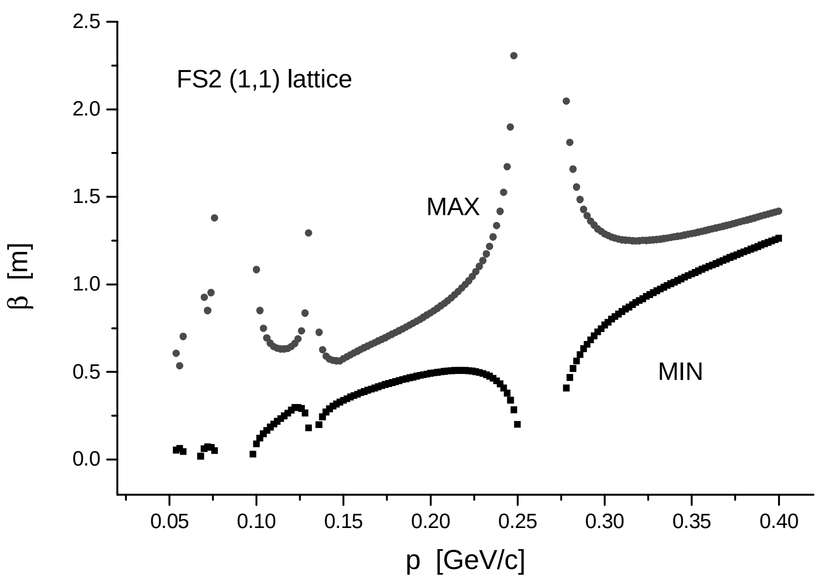

Figure 3. Minimum and maximum values of the matched beta function versus momentum.

We clearly see the π resonance around 260 MeV/c and the 2π resonance around 135 MeV/c. On the other hand there is no hint of a $3\pi/2$ half-integer resonance here. The value of the minimum beta function at 200 MeV/c is 49 cm.

It is interesting to note that, although there are two regions of good momentum transmission in this lattice, they have very different characteristics. Fig. 4 shows the dependence of the matched beta function on position across the cell for a sample momentum from regions I and II.

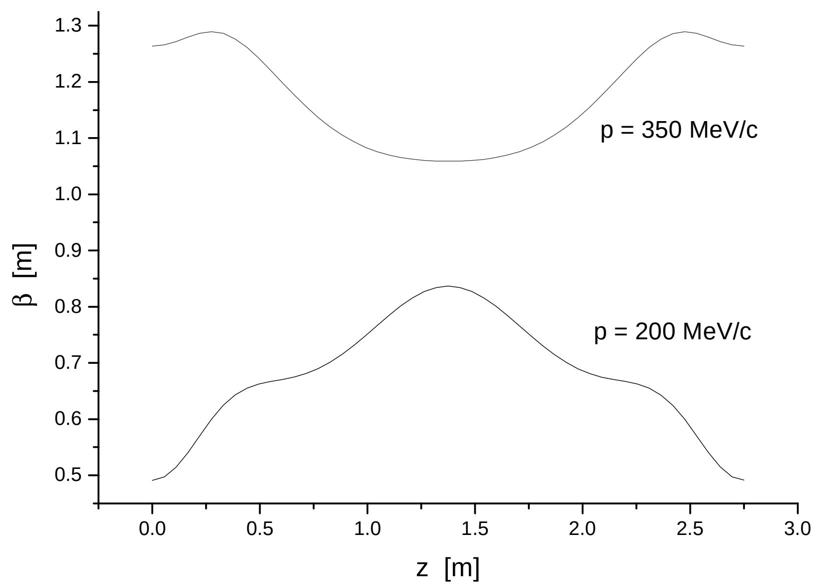

Figure 4. Beta function versus axial position. The solution is shown for two momenta corresponding to regions I and II.

Notice that the 200 MeV/c solution has a minimum in the beta function at the beginning of the cell. This is the location of the absorber in the FS2 cooling lattice, which must be in a region of low beta function. This configuration has been referred to as a SFOFO cooling lattice. However, if the lattice is left alone and we simply raise the beam momentum above the π resonance, the start of the cell corresponds roughly to the maximum in the beta function. The minimum occurs in the center of the cell where the magnetic field is near maximum. This configuration has been referred to as an ASOL cooling lattice. This behavior of the beta function has been noted previously [5,6].

Once the matched beta function has been determined for each momentum, the starting beam size and divergence can be computed for transmission studies using standard relations. The resulting values for σ_X and σ_Y are shown in Fig. 5.

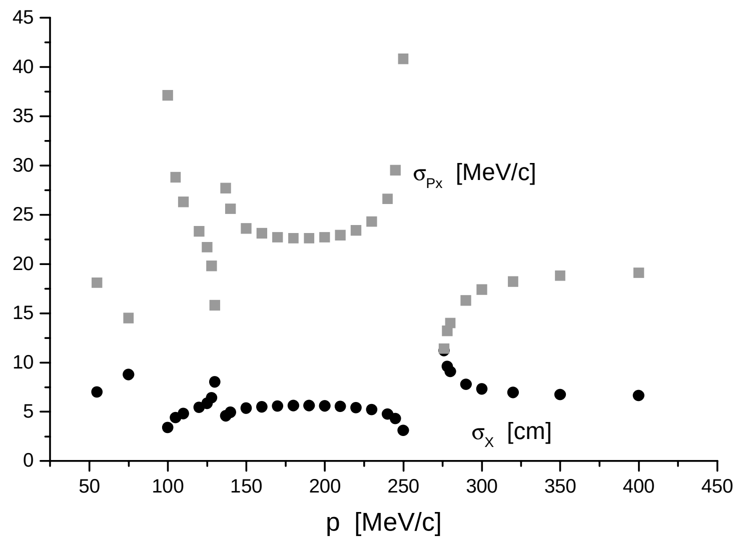

Figure 5. Initial values of matched beam size and divergence as a function of momentum.

The transmission behavior of the lattice with matched initial conditions is shown in Fig. 6.

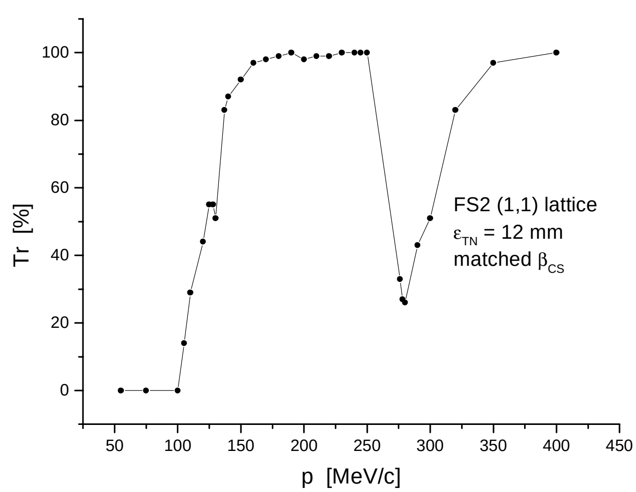

Figure 6. Transmission versus momentum for matched initial conditions.

The main features of Fig. 6 are the same as the unmatched plot in Fig. 2. The major difference is the transmission loss at the π resonance is less pronounced for the matched case. We were also concerned that part of the loss in transmission at low momentum may be due to the 1 m radial aperture used in the simulation and not due to resonance behavior. However, when we reexamined the low momentum cases with an aperture of 100 m, the results only changed by a few per cent.

5 Simple symplectic matrix analysis

One simple way of examining the stability of the lattice is to determine the eigenvectors of the one-cell transport matrix [9]. This method has been shown to give reasonable results for simple periodic solenoid lattices [10]. Here a calculation with lumped elements gave poor agreement with the tracking results. Instead we break the cell into N steps, each of which consists of a half drift, a thin lens and another half drift. This procedure is known to be symplectic [11].

For the solenoidal lattice considered here the focusing function K is given by

$$K(z) = \kappa^2(z) = \left(\frac{e B_Z(z)}{2 p_Z} \right)^2 \qquad \textbf{(9)}$$

The matrix for a thin lens focusing element is

$$F(K,d) = \begin{vmatrix} 1 & 0 \\ -K d & 1 \end{vmatrix} \qquad \textbf{(10)}$$

where K is the focusing strength and d is the length of the focusing element. The drift space matrix is given simply by

$$D(d) = \begin{vmatrix} 1 & d \\ 0 & 1 \end{vmatrix} \qquad \textbf{(11)}$$

where d is the length of the drift space. The transfer matrix M for one cell of the lattice can be found from

$$M = \left[\prod_{i=1}^{N} D(½ ds_i)\, F(K(s_i), ds_i)\, D(½ ds_i) \right] M_o \qquad \textbf{(12)}$$

where $M_o = I$ is the initial value of the transport matrix. We have the constraint

$$\sum_{i=1}^{N} ds_i = L \tag{13}$$

where L is the length of the cell. We used N = 275 for this calculation.

The trace of the matrix M is shown in Fig. 7.

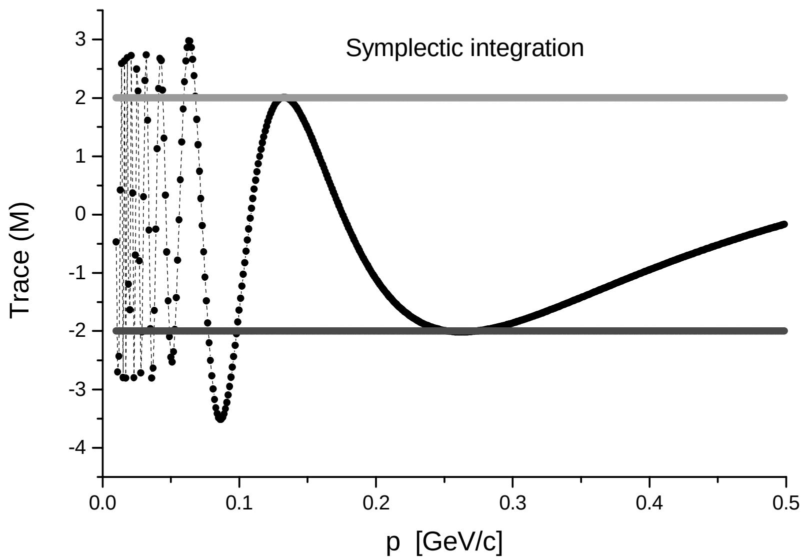

Figure 7. Trace of transport matrix from symplectic matrix theory.

The beam is unstable when the magnitude of the trace exceeds 2. This analysis predicts the π resonance lies around 260 MeV/c and that the 2π resonance occurs around 130 MeV/c.

The beta function can also be extracted from the same analysis because the transport matrix after one cell can be written in the form [9]

$$M = \begin{vmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{vmatrix} \tag{14}$$

The beta function from this analysis is shown in Fig. 8. The variation of the beta function with z across the cell was found by shifting the starting position of the matrix calculation to positions corresponding to different locations in the periodic magnetic field profile.

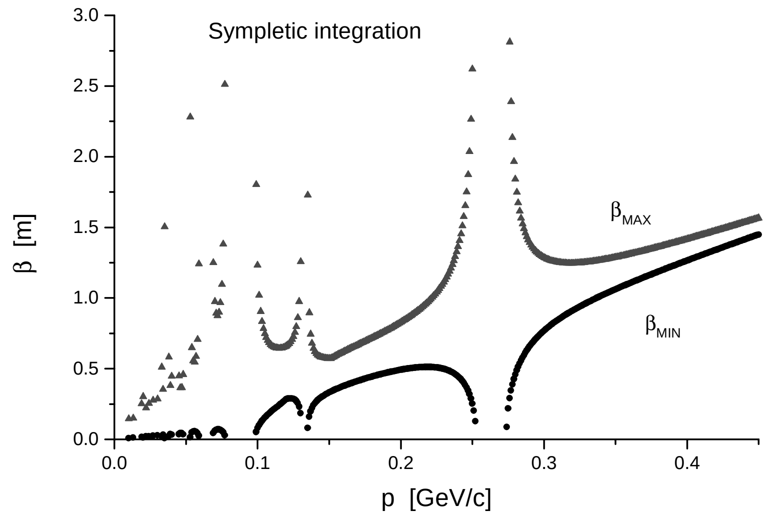

Figure 8. Beta function from symplectic matrix analysis.

This agrees very well with the matched beta function determined with the optimizer code.

6 Theory of Wang and Kim

The motion of particles in a periodic lattice is governed by Hill's equation [9].

$$\frac{d^2x(s)}{ds^2} + K(s)\, x(s) = 0 \tag{15}$$

Wang and Kim [6] have determined a recursive solution to this problem by expanding *K* in a complex Fourier series

$$\left(\frac{\lambda}{\pi}\right)^2 K(z) = \sum_{n=-\infty}^{\infty} \theta_n \exp(i\, 2\, n\, \pi\, z\, /\, \lambda) \tag{16}$$

Note that this is essentially the Fourier decomposition of the square of the lattice magnetic field.

Table 3 gives the values of the lowest order non-zero coefficients for the case of p = 200 MeV/c. For this analysis we used a total of 256 Fourier terms.

Table 3 Fourier coefficients	
n	θ_n
0	6.8483
2	0.5388
4	-0.0872
6	-1.6244
8	-1.2805
10	-0.7616
12	-0.2916
14	-0.0550
16	0.0293
18	0.0370
20	0.0220

Note that the coefficients are real numbers and that only even coefficients appear here since the focusing function is even. An important parameter in this theory is

$$\omega = \sqrt{\theta_o} \qquad \textbf{(17)}$$

The trace of the one period transport matrix was denoted Δ and given as

$$\Delta = \Delta_0 + \Delta_1 + \Delta_2 + \Delta_3 + \cdots \qquad \textbf{(18)}$$

The first three terms in the series are given by

$$\begin{aligned}\Delta_0 &= 2\cos\omega\pi \\ \Delta_1 &= 0 \\ \Delta_2 &= \frac{\pi \sin\omega\pi}{2\,\omega} \sum_{n=1}^{\infty} \frac{|\theta_n|^2}{\theta_0 - n^2}\end{aligned} \qquad \textbf{(19)}$$

The third order term Δ_3 was also used here and is given explicitly as Eq. 25 in reference 6. The computed value of Δ is given in Fig. 9.

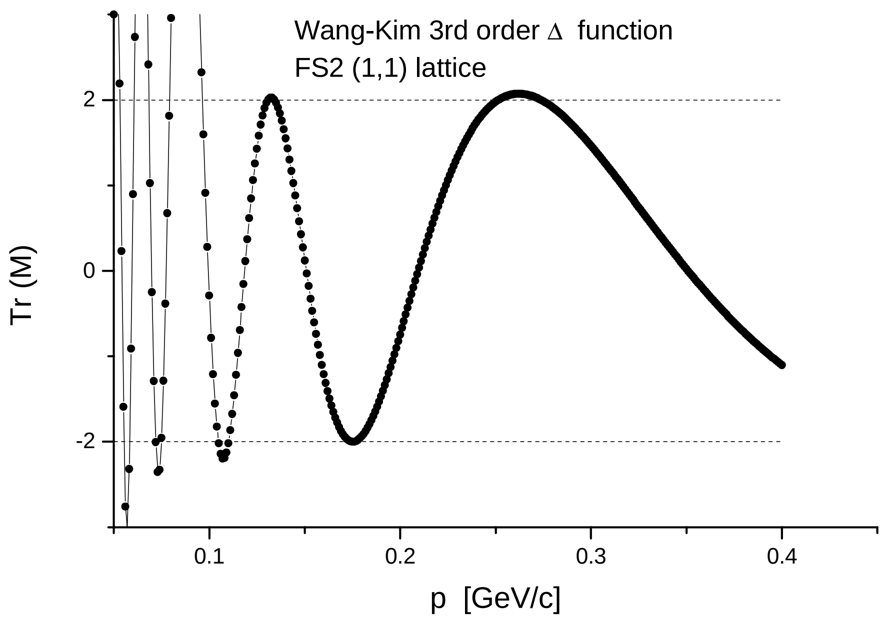

Figure 9. Trace of single period transport matrix from the analysis of Wang and Kim.

We see that the π resonance is predicted to occur around 260 MeV/c and the 2π resonance around 130 MeV/c. In contrast with the simple matrix analysis, there is also evidence here for a $3\pi/2$ half-integer resonance around 180 MeV/c.

Wang and Kim can also use their Fourier coefficients to compute the beta function for the lattice.

$$\beta(z) = \beta_0(z) + \beta_1(z) + \beta_2(z) + \cdots \tag{20}$$

The first two terms in the series are

$$\begin{aligned}\beta_0(z) &= \frac{\lambda \sin\omega\pi}{\pi\,\omega \sin\mu} \\ \beta_1(z) &= \frac{\lambda \sin\omega\pi}{\pi\,\omega \sin\mu} \sum_{n=1}^{\infty} \frac{Re[\theta_n \exp(i\,2n\pi z/\lambda)]}{n^2 - \theta_0}\end{aligned} \tag{21}$$

The second order term was also used and is given explicitly in Eq. 30 in reference 6. The quantity μ is the phase advance per period and is related to the stability parameter Δ by

$$\cos\mu = \frac{\Delta}{2} \tag{22}$$

Thus this prescription for finding the beta function is only valid in stable regions where $\Delta \le 2$. The beta function is shown in Fig. 10.

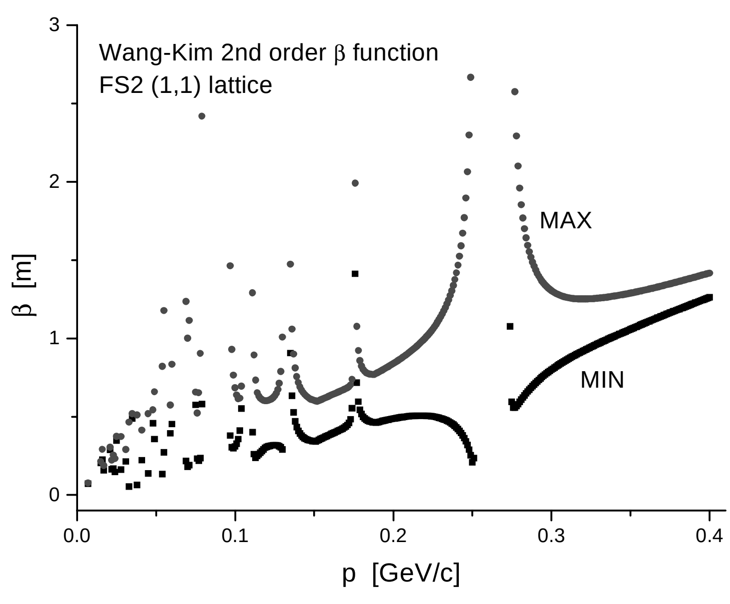

Figure 10. Beta function versus momenta using the Wang-Kim method.

This plot doesn't add anything new to the stability analysis since it is closely tied to the calculation of Δ. The computed value of the minimum beta at 200 MeV/c is 49 cm, in excellent agreement with the calculation mentioned earlier using the differential equation for β(z).

7 Conclusions

The FS2 (1,1) lattice has a rich harmonic structure. A summary of the observed resonance behavior is given in Table 4.

Table 4 Momentum range of stop bands [MeV/c]

	π	2π	3π/2
ICOOL	~255-320	~125-135	~200 (weak)
Balbekov (1,2)	252-318	134-152	none
optimizer β	252-275	131-134	none
symplectic	253-273	131-134	none
Wang & Kim	252-272	131-134	175

All of these methods agree pretty well with the tracking results. For the ICOOL simulation the width of the π resonance is the range where the transmission of the matched curve is below 80%. For the optimizer determination of the β function we defined the resonance regions to have χ^2 greater than 10^{-3} or α at the end of the cell greater than 10^{-3}. The Hill function analysis of Wang & Kim was the only method that predicted a half-integer resonance.

Acknowledgments

We would like to thank Scott Berg and Bob Palmer for useful discussions.

Notes and references

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