Calculations Of Predicted Performance For μ-Cooling Rings

David Neuffer
Fermilab

Abstract

In this note we present the analytical equations for ionization cooling performance and apply them to recently developed storage ring cooling systems designed to cool muons transversely and longitudinally. Expected cooling properties are calculated and variations that may improve cooling performance are discussed.

Introduction

Recently A. Garren, H. Kirk, and Y. Fukui[1] have initiated a study of a ring cooler designed to cool both longitudinally and transversely, using quadrupoles for focusing and hydrogen wedge absorbers for cooling.

In this paper we present some calculations of cooling parameters, rates, and equilibrium emittances for this initial cooling ring.

Energy Cooling equations

The basic equations for transverse and longitudinal equations have been presented in previous references. The differential equation for rms transverse cooling is [2, 3, 4, 5, 6, 7]:

$$\frac{d\varepsilon_N}{ds} = -\frac{1}{\beta^2 E}\frac{dE}{ds}\varepsilon_N + \frac{\beta\gamma\ \beta_\perp}{2}\frac{d\langle\theta_{rms}{}^2\rangle}{ds} = -\frac{1}{\beta^2 E}\frac{dE}{ds}\varepsilon_N + \frac{\beta_\perp E_s^2}{2\beta^3 m_\mu c^2 L_R E} \qquad (1)$$

where the first term is the energy-loss cooling effect and the second is the multiple scattering heating term. Here ε_N is the normalized transverse emittance, E is the beam energy, $\beta = v/c$ and γ are the usual kinematic factors, dE/ds is the energy loss rate, θ_{rms} is the rms multiple scattering angle, L_R is the material radiation length, $\beta_\perp$ is the betatron function, and E_s is the characteristic scattering energy (~13.6 MeV).[7] (The normalized emittance is related to the geometric emittance $\varepsilon_\perp$ by $\varepsilon_N = \varepsilon_\perp/(\beta\gamma)$, and the beam size is given by $\sigma_x = (\varepsilon_\perp\beta_\perp)^{1/2}$.)

The equation for longitudinal cooling with energy loss is:

$$\frac{d\sigma_E^2}{ds} = -2\frac{\partial\frac{dE}{ds}}{\partial E}\sigma_E^2 + \frac{d\langle\Delta E_{rms}{}^2\rangle}{ds} \qquad (2)$$

in which the first term is the “cooling” term and the second is the heating term caused by random fluctuations in the particle energy. Beam cooling can occur if the derivative $\partial(dE/ds)/\partial E > 0$. This energy loss can be estimated by the Bethe-Bloch equation[8]:

$$\frac{dE}{ds} = 4\pi N_A r_e^{\ 2} m_e c^2 \rho \ \frac{Z}{A}\left[\frac{1}{\beta^2}\ln\left(\frac{2m_e c^2\gamma^2\beta^2}{I}\right) - 1 - \frac{\delta}{2\beta^2}\right] \tag{3}$$

where N_A is Avogadro's number, ρ, A and Z are the density, atomic weight and number of the absorbing material, m_e and r_e are the mass and classical radius of the electron, ($4\pi N_A r_e^{\ 2} m_e c^2 = 0.3071$ MeV cm^2/gm). The ionization constant I is approximately 16 $Z^{0.9}$ eV, and δ is the density effect factor which is small for low-energy μ′s. (δ = 0 is used in this paper.) The energy loss as a function of p_μ is shown in Fig. 1. The derivative is negative (or naturally heating) for $E_\mu < \sim 0.3$ GeV, and is only slightly positive (cooling) for higher energies.

In the long-pathlength Gaussian-distribution limit, the second term in Eq. 2 is given approximately by[9]:

$$\frac{d\left\langle \Delta E_{rms}^{\ 2}\right\rangle}{ds} = 4\pi\left(r_e m_e c^2\right)^2 n_e \gamma^2\left(1 - \frac{\beta^2}{2}\right) \cong 0.157\rho\frac{Z}{A}\gamma^2\left(1-\frac{\beta^2}{2}\right)\ (\mathrm{MeV})^2\mathrm{cm}^2/\mathrm{gm}, \tag{4}$$

where n_e is the electron density in the material ($n_e = N_A\rho Z/A$). This expression increases rapidly with higher energy (larger γ), opposing the cooling process. After adding this energy straggling, ionization cooling does not naturally provide adequate longitudinal cooling.

However, the cooling term can be enhanced by placing the absorbers where transverse position depends upon energy (nonzero dispersion) and where the absorber density or thickness also depends upon energy, such as in a wedge absorber.(see fig. 4-3) In that case the cooling derivative can be rewritten as:

$$\frac{\partial \frac{dE}{ds}}{\partial E} \Rightarrow \left.\frac{\partial \frac{dE}{ds}}{\partial E}\right|_0 + \frac{dE}{ds}\frac{\eta\rho'}{\beta c p \rho_0} \tag{5}$$

where ρ'/ρ_0 is the relative change in density with respect to transverse position, ρ_0 is the reference density associated with dE/ds, and η is the dispersion ($\eta = d\,x/d(\Delta p/p)$). Increasing the longitudinal cooling rate in this manner decreases the transverse cooling by the same amount. The transverse cooling term is changed to:

$$\frac{d\varepsilon_N}{ds} = -\frac{1}{\beta^2 E}\frac{dE}{ds}\left(1 - \frac{\eta\rho'}{\rho_0}\right)\varepsilon_N \tag{6}$$

Note that the coupled transverse cooling (and heating) changes occur in the same direction (i.e. horizontal or vertical) as the dispersion and wedge. However the sum of the cooling rates (over x, y, and z) remains constant. This sum can be represented, as with radiation damping, as a sum of cooling partition numbers, where the partition number is defined as the ratio of the cooling rate to the fractional momentum loss rate. For x and y emittance cooling the partition numbers are both naturally 1:

$$g_y = g_x = \frac{\frac{d\varepsilon_x/ds}{\varepsilon_x}}{\frac{dp/ds}{p}} = 1. \tag{7}$$

The partition number for longitudinal cooling is given by

$$g_L = \frac{\frac{d\varepsilon_L/ds}{\varepsilon_L}}{\frac{dp/ds}{p}} = \frac{\partial(dE/ds)}{\partial E} \Bigg/ \frac{\frac{dp}{ds}}{p} = \frac{\frac{\partial(dp/dt)}{\partial p}}{\frac{(dp/dt)}{p}} \quad . \tag{8}$$

This can be evaluated for the Bethe-Bloch dE/ds formula (with $\delta = 0$, and no dispersion and wedge absorbers) as:

$$g_{L,0} = \frac{-2\,\mathrm{Ln}[K(\gamma^2-1)] + 2\gamma^2}{\gamma^2\mathrm{Ln}[K(\gamma^2-1)] - (\gamma^2-1)} \tag{9}$$

where $K = 2m_ec^2/I$. With the wedge at nonzero dispersion, $g_L=g_{L,0} + \eta\rho'/\rho_0$, and the transverse partition number in the wedge plane g_x becomes:

$$g_x = 1 - \frac{\eta\rho'}{\rho_0} \tag{10}$$

The sum of the partition numbers $\Sigma_g = (g_x + g_y + g_L)$ is a function of muon momentum, and is displayed in figure 2. Σ_g is approximately 2 for $P_\mu > 0.3$ GeV/c, but is smaller for lower energies.

In longitudinal phase space, there are two rms heating terms to be considered:

1. The energy straggling term shown in eq. 4 (which is proportional to γ^2, and therefore larger for higher-energy beams), and
2. the anti-damping due to a negative partition number, which occurs at low momenta ($P_\mu <$ ~0.35 GeV) where the Bethe-Bloch formula has a negative derivative.

Effective longitudinal cooling requires opposing the sum of these heating effects with dispersion-wedge absorbers. We hypothesize that the task of energy cooling would be relatively easy at energies where the sum of these is minimized. The relative magnitude of these depends on the beam energy spread; however in Fig. 3 we display the two terms at $\Delta E_\mu = 20$ MeV (for Be absorbers), a typical muon source energy spread. The sum is minimized at $P_\mu \cong 400$ MeV/c, within a relatively broad range of $P_\mu \cong 300$ to 500 MeV/c. The heating terms do become dramatically larger outside that range, for both smaller and larger momenta.

Longitudinal Betatron Functions

The longitudinal equations of motion for muons in a storage ring are:

$$\frac{d\Delta E}{ds} = eV'(\cos(\phi+\phi_s) - \cos\phi_s) \cong -eV'\sin\phi_s\,\phi$$
$$\frac{d\phi}{ds} \cong \frac{1}{\beta^3\gamma}\left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}\right)\frac{2\pi}{\lambda_0}\frac{\Delta E}{mc^2} = \frac{1}{\beta^3\gamma}\alpha_p\frac{2\pi}{\lambda_0}\frac{\Delta E}{mc^2} \tag{11}$$

Energy offset (ΔE) and particle rf phase (ϕ) are used as variables. In these equations, $V' = V_{rf}/C_{ring}$ is the average acceleration gradient in the ring, λ_0 is the rf wavelength, ϕ_s is the rf phase of the reference bunch particle. ($V' \cos\phi_s$ is the mean energy loss rate in the absorber: $V' \cos\phi_s = dE/ds\ L_{absorbers}/C_{ring}$.) Also ($\alpha_p=1/\gamma^2-1/\gamma_t^2$) is the frequency slip factor. From these coupled equations, we can define a longitudinal "betatron function" β_ϕ as:

$$\frac{\langle\phi^2\rangle}{\langle\Delta E^2\rangle} = \frac{1}{\beta^3\gamma\ eV'\sin\phi_s}\frac{2\pi}{\lambda_0}\frac{\alpha_p}{mc^2} \equiv \beta_\phi^{\ 2} \tag{12}$$

Assuming the longitudinal motion is near-equilibrium ($\langle\phi\ \Delta E\rangle = 0$), the longitudinal rms quantities are:

$$\varepsilon_L = \sqrt{\langle\phi^2\rangle\langle(\Delta E)^2\rangle}\ ;\ \langle\phi^2\rangle = \varepsilon_L\beta_\phi\ ;\ \langle(\Delta E)^2\rangle = \varepsilon_L/\beta_\phi\ . \tag{13}$$

This obtains an invariant emittance in energy units (i. e., MeV). Changing to length units requires multiplying by the wave number ($\lambda_0/2\pi$) and dividing by the muon rest energy (105.66 MeV/c).

With these parameters, we can now define a cooling equation in longitudinal emittance. This is simply:

$$\frac{d\varepsilon_L}{ds} = -\frac{g_L}{\beta^2 E}\frac{dE}{ds}\varepsilon_L + \frac{\beta_\phi}{2}\frac{d\langle\Delta E_{rms}{}^2\rangle}{ds} \tag{14}$$

From this equation an expression for the equilibrium longitudinal emittance $\varepsilon_{L,eq}$ can be obtained:

$$\varepsilon_{L,eq} = \frac{\beta^2 E}{dE/ds}\frac{\beta_\phi}{2g_L}\frac{d\langle\Delta E_{rms}{}^2\rangle}{ds} \tag{15}$$

Similarly an expression for the equilibrium transverse emittance can be obtained:

$$\varepsilon_{x,N,eq} = \frac{1}{g_x dE/ds}\frac{\beta_\perp E_s^2}{2\beta\ m_\mu c^2 L_R} \tag{16}$$

RF Bucket Acceptance

Another important criteria for a storage ring is the longitudinal acceptance (rf bucket size) of the rf system. From the equations of longitudinal motion a stable region is defined by the equation of the separatrix:

$$\frac{(\Delta E)^2}{2} = \frac{eV'\lambda_0 mc^2\beta^3\gamma}{2\pi\alpha_p}\left[(\phi_o - \phi)\cos\phi_s + (\sin(\phi + \phi_s) - \sin(\phi_0 + \phi_s))\right] \qquad (17)$$

with $\phi_0 = -2\,\phi_s$ defining the separatrix boundary. The limits in energy spread are defined by:

$$\Delta E = \pm\sqrt{\frac{eV'\lambda_0 mc^2\beta^3\gamma}{\pi\alpha_p}\left[(-2\phi_s)\cos\phi_s + 2\sin(\phi_s)\right]}\ . \qquad (18)$$

This acceptance can become very restricted for lower energy muon beams. For the parameters of Table 1, ΔE = 36 MeV, 100 MeV and 300 MeV are obtained for 250, 500, and 1000 MeV/c beams, respectively. The rf bucket for the 250 MeV/c beam is uncomfortably small compared to the injected beam sizes.

Example: Application to the Garren-Kirk cooling ring

We can now apply the parameters of the Garren-Kirk cooling ring to these formulae to develop an understanding of expected cooling performance at its parameters. In initial studies the beam central momentum was set to P_μ = 250, 500 and 1000 MeV/c. The ring lattice has a cell length of L_{cell}=10.55m, with 8 cells required to complete a full turn. Cell lattice parameters are shown graphically in fig. WWWW.

The lattice includes a liquid hydrogen absorber of length 25cm (dE/ds=0.3 MeV/cm) with wedges at the entrance and exit of $\theta = 10°$. The dispersion η at the absorber is 45cm, while the betatron function $\beta_\perp$ is 25 cm at the center of the absorber (mean value of 27cm in the absorber). The total rf per cell is 20 MV, so the equilibrium phase is obtained from $\cos(\phi_s) = 7.5/20$, which implies $\sin(\phi_s) = 0.927$. Also γ_t =32.1 in this lattice. γ_t is a critical parameter in setting the properties of the longitudinal motion. The rf bucket is relatively restricted in the case with P_μ = 250MeV/c, where the rf bucket extends over $\Delta E = \pm 35$MeV. The rf bucket acceptance is $\Delta E = \pm 100$MeV for P_μ = 500MeV/c and $\Delta E = \pm 290$MeV for P_μ = 1000MeV/c.

In emittance exchange calculations a critical parameter is the change in partition number due to the wedges. That factor for our example is:

$$\Delta g_L = \frac{\eta\rho'}{\rho_0} = \eta \frac{2\tan\theta}{L_{absorber}}, \qquad (19)$$

which is 0.635 at these parameters. This implies that g_x is reduced to ~0.365 in our baseline example. This means that the equilibrium x-emittance is almost three times larger than the equilibrium y-emittance. In some early simulations that equilibrium emittance is larger than the acceptance; this means that the ring would have large transverse losses.

Table 1 displays some key cooling parameters for the parameters of the Garren-Kirk cooling ring; the table displays parameters for P_μ = 250, 500, and 1000 MeV/c. The cooling performance under ICOOL simulations closely

Discussion

Some guidelines for expected performance and future design improvements can be obtained from these evaluations.

At 250MeV/c, the unperturbed partition number $g_{L,0}$ is negative (~ -0.16), or anti-damping, which increases the difficulty of cooling. This difficulty increases at lower momenta (at 200 MeV/c $g_{L,0}$ is ~ -0.3 and at 150 MeV/c it is ~-0.5). This anti-damping is also somewhat exacerbated by the energy spread; lower energy particles in the distribution receive relatively large anti-damping and can be lost before synchrotron oscillations return the particles to higher energies. As discussed above, longitudinal cooling is probably optimal at slightly higher energies.

In Table 1 we have assumed the dispersion/wedges are horizontal and x and y motions are decoupled. A lattice with x-y rotation between bends (solenoid or skew-quad) could exchange x and y motions between wedges and therefore share the partition number changes equally between x and y. This may be suggested for future studies, particularly since the anti-damping places the horizontal beam size ("cooled" equilibrium) quite large, and actually larger than the ring acceptance in an initial lattice.[1]

For optimum acceptance the beam should be matched to the ring properties when injected into the ring. (Losses due to longitudinal mismatch are seen in some early examples [1].) In particular bunch length and energy spreads must be matched into the stable injection bucket. The factor β_ϕ identifies the matched ration of bunch length to energy spread. (see eq. 12 and Table 1.) For example, at the parameters of Table 1, a bunch with rms energy spread δE of 20 MeV would have a matched rms bunch length ($c\tau$) of 19.6, 9.9 and 2.34 cm for 250, 500 and 1000 MeV/c beams, respectively.

The initial simulation results support the rms analysis discussed here. Future versions of ring coolers can use these rms formulae as guidelines to ensure that the cooling rings have adequate longitudinal and transverse acceptances for matched cooling and that the injected beams are matched to the cooling ring parameters.

Table 1: Cooling Parameters for the sample cooling ring							
$\mathbf{P_\mu}$	$\mathbf{g_{L,wedge}}$	$\mathbf{g_{L,0}}$	$\mathbf{d(\Delta E^2)/ds}$	β_ϕ	$\varepsilon_{y.eq.}$	$\varepsilon_{L,eq}$	$\varepsilon_{x.eq.}$
MeV/c			MeV^2/m	MeV^{-1}	cm	cm	cm
250	0.635	-0.160	0.042	0.0411	0.103	0.31	0.282
500	0.635	0.0604	0.136	0.0144	0.099	0.52	0.271
1000	0.635	0.114	0.509	0.0049	0.091	1.18	0.250

References

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Figure 1.
dE/dx for various materials from ref. (). Note that the derivative of dE/dx with respect to P_μ is strongly negative (naturally heating) for $P_\mu < \sim 400$ MeV/c and is only slightly positive for larger P_μ.

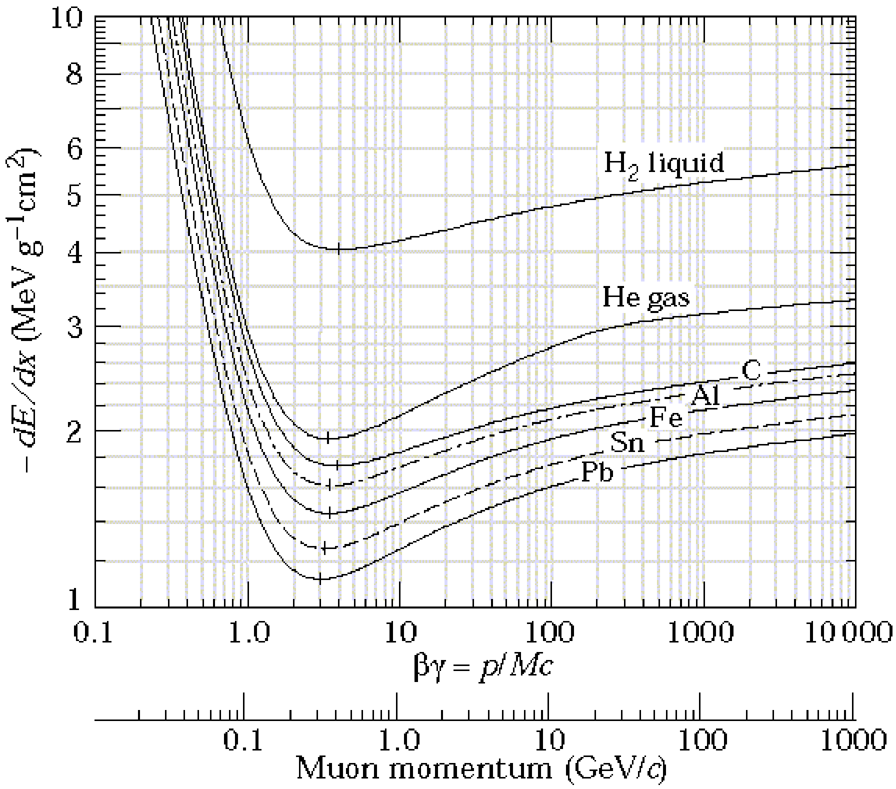

Figure 2. Partition numbers (in Be) as a function of muon momentum P_μ. Σ_g is the sum of x, y and z partition numbers, $g_L = g_z = \Sigma_g - 2$ is the longitudinal partition number.

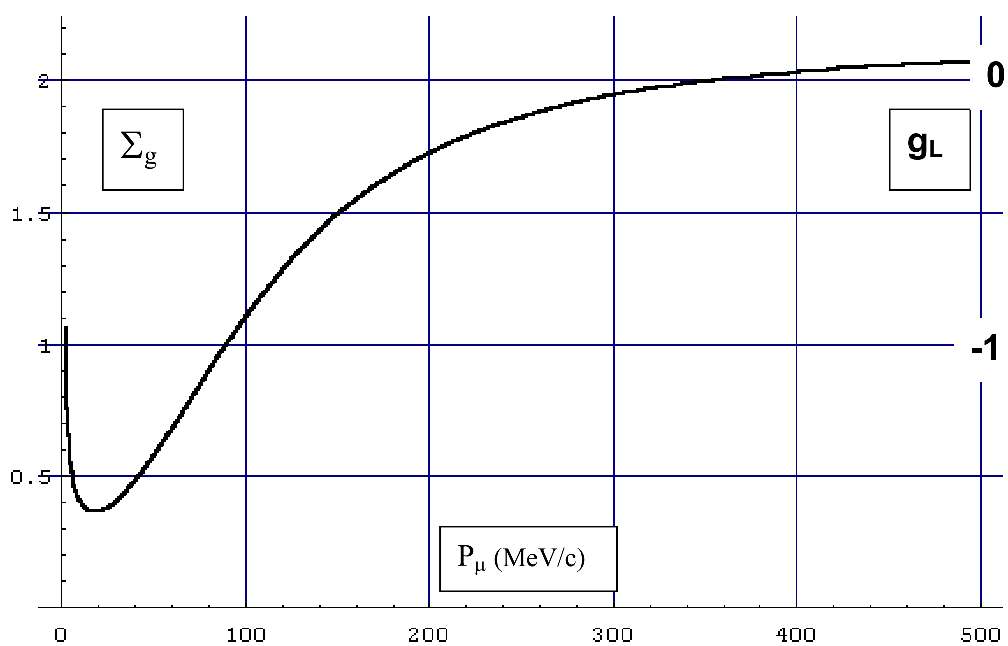

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Figure 3: Energy-spread heating terms in ionization cooling.

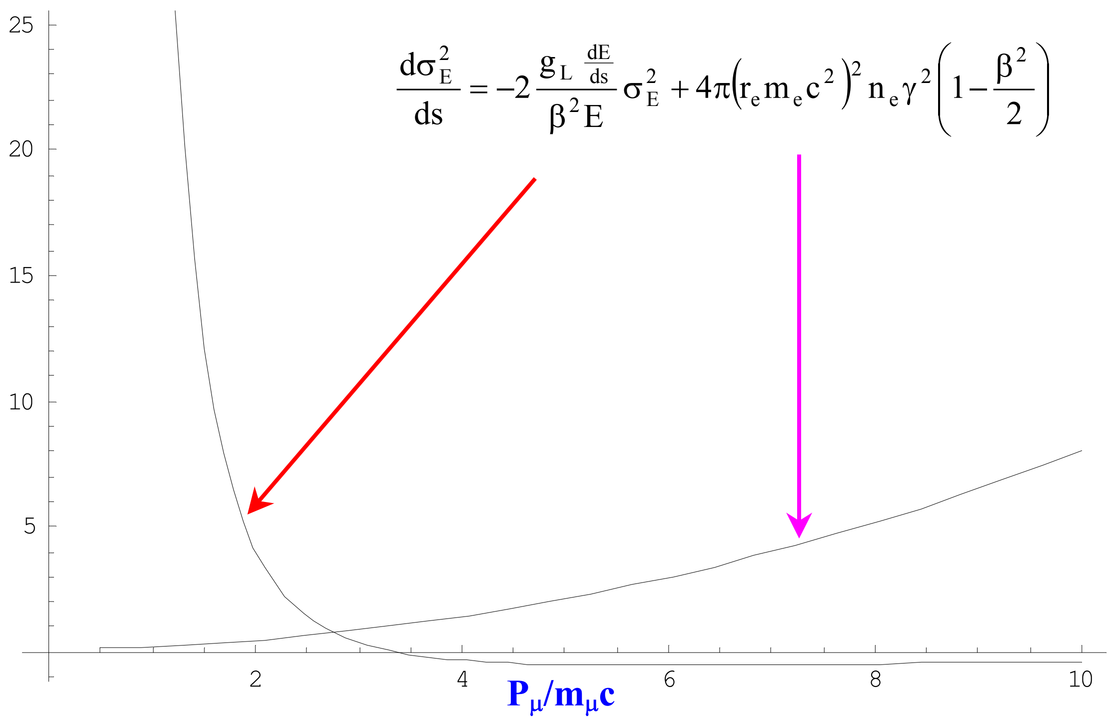

Figure 4. Lattice functions for a cell of a cooling lattice.